Obstruction theory

25 November 2020 15:00

Question:
- $A \subset X$
- $A \xrightarrow{f} Y$
- When does it possible to extend f to the map $g: X \to Y$, i.e $g|_A = f$.

Plan:

1) X - CW complex, A - n-th skeleton
2) Fibrations. „Primiry charact. class"
3) X - CW complex $\quad H^n(X;\pi) \longleftrightarrow [X, K(\pi,n)]$
4) Hopf theorem

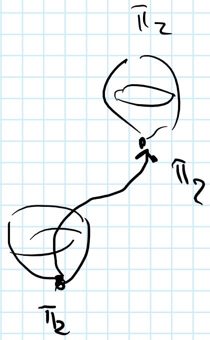
① X - CW complex

X^n - n-th skeleton of X

Y - homotopicaly simple = no action of π_1 on π_i $i>1$

Start with $f: X^n \to Y \quad \underset{\text{whant}}{\rightsquigarrow} \quad g: X^{n+1} \to Y \quad g|_{X^n} = f$.

$$\begin{tikzcd} S^n \arrow[r, "h"] \arrow[d, hook] & X^n \arrow[r, "f"] \arrow[d, hook] & Y \\ D^{n+1}_e \arrow[r, "h"] & X & \end{tikzcd} \quad \rightsquigarrow \quad \begin{array}{l} S^n \xrightarrow{h} X^n \xrightarrow{f} Y \\ [f \circ h] \text{ in } [S^n, Y] \\ \qquad\qquad\quad \pi_n(Y) \end{array}$$



Each e - n+1 cell in $X \rightsquigarrow$ a class in $\pi_n(Y)$

$c_f \in C^{n+1}_{CW}(X; \pi_n(Y))$

Stupid theorem: $c_f = 0 \iff \exists$ such g

Theorem: 1) $\partial c_f = 0 \Rightarrow$ define $C_f := [c_f] \in H^{n+1}(X; \pi_n(Y))$

2) $C_f = 0 \iff \exists g: X^{n+1} \to Y$ such that $g|_{X^{n-1}} = f|_{X^{n-1}}$

Proove 1)

$$\begin{tikzcd} C_{n+2}(X) = H_{n+2}(X^{n+2}, X^{n+1}) \arrow[r, "h^{-1}"] \arrow[dd, "\partial"] & \pi_{n+2}(X^{n+2}, X^{n+1}) \arrow[d, "\partial"] \\ & \pi_{n+1}(X^{n+1}) \arrow[d] \\ C_{n+1}(X) = H_{n+1}(X^{n+1}, X^n) \arrow[r, "h^{-1}"] \arrow[ddr, "c_f"] & \pi_{n+1}(X^{n+1}, X^n) \arrow[d, "\partial"] \\ & \pi_n(X^n) \arrow[d, "f_*"] \\ & \pi_n(Y) \end{tikzcd}$$

$\left.\begin{array}{l}\pi_{n+1}(X^{n+1}) \\ \pi_{n+1}(X^{n+1}, X^n) \\ \pi_n(X^n)\end{array}\right\} 0$

Proove 1)

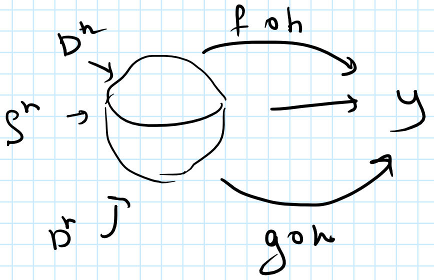
$$\begin{tikzcd}
\mathcal{C}_{n+2}(X) = H_{n+2}(X^{n+2}, X^{n+1}) \arrow[r, "h^{-1}"] \arrow[dd, "\partial"] & \pi_{n+2}(X^{n+2}, X^{n+1}) \arrow[d, "\partial"] \\
 & \pi_{n+1}(X^{n+1}) \arrow[d] \\
\mathcal{C}_{n+1}(X) = H_{n+1}(X^{n+1}, X^{n}) \arrow[r, "h^{-1}"] \arrow[ddr, "c_f"'] & \pi_{n+1}(X^{n+1}, X^{n}) \arrow[d, "\partial"] \\
 & \pi_n(X^n) \arrow[d, "f_*"] \\
 & \pi_n(Y)
\end{tikzcd}$$

2) Def Difference cochain $d_{f,g} \in C^n(X; \pi_n(Y))$

$$\begin{tikzcd} X^n \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd} \qquad f|_{X^{n-1}} = g|_{X^{n-1}}$$

e – n-cell of X $\quad h: D^n \to X \qquad e \rightsquigarrow [k_e]$

$k_e : S^n \to Y$



What this means?

- $d_{f,g} = 0 \iff f \sim g$ fixed on X^{n-1}
- If $f, g : X \to Y$ we can use homotopy to make f agree with g on X^n

Properties:

1) $\partial d_{f,g} = c_g - c_f$

2) For any $f : X^n \to Y$ $\rightsquigarrow$ $\exists\, g : X^n \to Y$, $g|_{X^{n-1}} = f|_{X^{n-1}}$
For any given $d \in C^n(X; \pi_n(Y))$ and $d = d_{f,g}$

3) $d_{g,f} = -d_{f,g}$

4) $d_{f,g} = d_{f,h} + d_{h,g}$

Proof of part 2

Suppose $c_f = 0 \implies \exists\, d : c_f = \partial d$

By 2) $\exists\, g : X^n \to Y$, $g|_{X^{n-1}} = f|_{X^{n-1}}$ $\quad d_{f,g} = -d$

By 1) $c_g = c_f + \partial d_{f,g} = \partial d - \partial d = 0 \Rightarrow g$ can be extended

Relative case: $f: A \cup X^n \to Y$ A subcomplex of X

Whant extention to $A \cup X^{n+1}$.

Theorem $f, g : X \to Y$ wich agree on X^{n-1}

Then 1) $d_{f,g}$ is a cocycle

2) $D_{f,g} = [d_{f,g}] \in H^n(X; \pi_n(Y))$ $D_{f,g} = 0 \iff f|_{X^n}$ and $g|_{X^n}$ are homotopic relative X^{n-1}

Applications:

I $\begin{matrix} F \to E \\ \downarrow p \\ B \end{matrix}$

- B simply conn / fibration is homotop-trivial
- F homotopy syple ...... "Y"
- B - CW complex "X"

Given a section $s: B^n \to E \xrightarrow{?} \bar{s}: B^{n+1} \to E$ $\bar{s}|_{B^n} = s$

$$\begin{tikzcd} & D^{n+1} \times F \arrow[r] \arrow[d] & E \arrow[d] \\ S^n \arrow[r, hook] \arrow[ur, bend left, "s"] & D^{n+1} \arrow[r, "h"] & B^n \subset B \end{tikzcd}$$

For each e: $S^n \to D^{n+1} \times F$

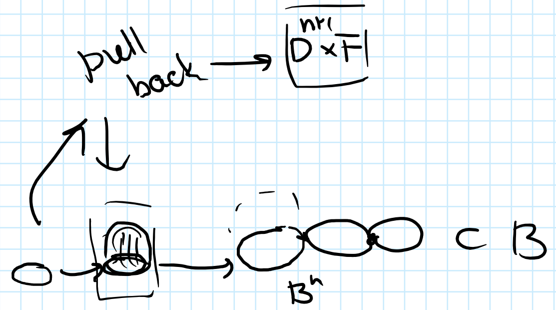
clas in $\pi_n(D^{n+1} \times F) = \pi_n(F)$

$\rightsquigarrow$ cochain $c_s \in C^{n+1}(B; \pi_n(F))$

Stupid theorem: $c_s = 0 \iff$ exestention exists

Theorem: 1) $\partial c_s = 0 \Rightarrow$ define $e_p = [c_p] \in H^{n+1}(B; \pi_n(F))$

2) $e_s = 0 \iff s$ can be changed on B^n, keeping on B^{n-1}, and extended to B^{n+1}



$X \xrightarrow{f} Y \rightsquigarrow$ $\begin{tikzcd} X \times Y \arrow[d] \\ X \arrow[u, bend right, "s"'] \end{tikzcd}$ $(x, f(x))$, x

$$\begin{array}{c} \downarrow \quad \uparrow s \\ X \quad x \end{array}$$

3) Suppose $\pi_0(F) = \ldots = \pi_{n-1}(F) = 0$, $\pi_n(F) \neq 0$

E $\Rightarrow$ always $\exists\, s: B^n \to E$

$\downarrow$ first non-trivial obstruction is ~~to~~ extention to B^{n+1}

B $\quad \ldots B^2 \supset B^1 \supset B^0$ Suppose $\exists\, s, s' : B^n \to E$

<u>Theorem</u> $C := C_s = C_{s'} \in H^{n+1}(B; \pi_n(F))$ $\leftarrow$ invariant E

„primary characteristic class"

- Naturality

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B \end{array}$$

- $C(f^*(E)) = f^* C(E)$
- f homotopy equiv. $-$ f^* iso cohomology

II Eilenberg - Maclane spaces

$K(\pi, n)$ $-$ any space; $\pi_i(K(\pi,n)) = 0$ $i \neq n$

$\pi_n(K(\pi,n)) = \pi$ $\quad$ (π is abelian)

$\pi = \{ g_i \; i \in I \mid$ relations $\}$

generators $\quad$ „$\sum k_I g_i = 0$"

$K(\pi, n) = \bigvee S_i^n$ + $n+1$ cell for each relation + $\geq n+2$ cells to kill all other homotopy groups

„generators" $\quad [\partial : e_i] = k_i$

<u>Theorem</u> X - CW complex $[X, K(\pi,n)] \overset{\text{bij}}{\longleftrightarrow} H^n(X; \pi)$

$K(\mathbb{Z}, 1) = S^1$ $\quad H^1(X; \mathbb{Z}) \longleftrightarrow [X, S^1]$

<u>Step 1</u> Find a „fundamental class" in $H^n(K(\pi,n); \pi)$
F

<u>Step 2</u> $[X \xrightarrow{f} K(\pi,n)] \longleftrightarrow [f^* F]$

Injectivity: $f, g: f^*F = g^*F \rightsquigarrow$ would like to construct a homotopy

Surjectivity: class $d \rightsquigarrow$ construct a map

Step 1 Algebraic: $H^n(K(\pi,n);\pi) = \mathrm{Hom}(H_n(K(\pi,n));\pi) = \mathrm{Hom}(\pi,\pi)$

Univ coef th

$H_n(K(\pi,n)) = \pi$ by H.

$F \dashrightarrow \mathrm{id}$

Geometric: Exersise

"X" $\xrightarrow{\text{const}}$ "Y"

Claim $F := d_{\text{const},\,\mathrm{id}}$ $K(\pi,n) \xrightarrow{\mathrm{id}} K(\pi,n)$

III Hopf theorem: X - n-dim, CW complex

$H^n(X;\mathbb{Z}) \longleftrightarrow [X, S^n]$

$K(\mathbb{Z},n) \neq S^n$ $\quad \underline{K(\mathbb{Z},n)} = \underbrace{\underline{S}^n + \cancel{X^{n+1}}} + \text{higher dim}$

Reference: Fomenko Fuchs Homotopy theory